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$\bar{\partial}$-PROBLEMS AND SOME APPLICATIONS

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0. Preliminaries.

Let D be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. We denote by $C^{1,\infty}(\partial D\times D)$ the space of all functions in $C^1(\partial D\times D)$ which are C^∞ in the second variable. A $(1,0)$-form $W=\sum_{j=1}^n w_j(\zeta,z)d\zeta_j$ is called a generating form with coefficients in $C^{1,\infty}(\partial D\times D)$ if W satisfies the following conditions (1) and (2):

(1) $w_j(\zeta,z)\in C^{1,\infty}(\partial D\times D)$.

(2) $\sum_{j=1}^n w_j(\zeta,z)(\zeta_j-z_j)=1$.

We define

$$\beta=|\zeta-z|^2,\quad B=\frac{\partial_\zeta\beta}{\beta},\quad I=[0,1].$$

The homotopy form on $(\partial D\times I)\times D$ associated to W is defined by

$$\hat{W}(\zeta,\lambda,z)=\lambda W(\zeta,z)+(1-\lambda)B(\zeta,z).$$

Cauchy-Fantappiè kernel $\Omega_q(\hat{W})$ of order q generated by $\hat{W}$ is defined by

$$\Omega_q(\hat{W})=\frac{(-1)^{q(q-1)/2}}{(2\pi i)^n}\binom{n-1}{q}\hat{W}\wedge(\bar{\partial}_{\zeta,\lambda}\hat{W})^{n-q-1}\wedge(\bar{\partial}_z\hat{W})^q,\quad 0\le q\le n-1.$$

$\Omega_q(W)$ is defined in the same way, with W instead of $\hat{W}$. We define $K_q=\Omega_q(B)$. Then we have the Cauchy-Fantappiè integral formula(cf. Range[22]):

Theorem 1. *For $1\le q\le n$, define the linear operator*

$$T_q^W:C_{0,q}(\overline{D})\to C_{0,q-1}(D)$$

by

$$T_q^Wf=\int_{\partial D\times I}f\wedge\Omega_{q-1}(\hat{W})-\int_D f\wedge K_{q-1},$$

and set $T_0^W=T_{n+1}^W=0$. Then the following holds:

(a) *For $k=0,1,\cdots,\infty$, if $f\in C_{0,q}^k(D)\cap C_{0,q}(\overline{D})$, then $T_q^Wf\in C_{0,q-1}^k(D)$.*

(b) *For $0\le q\le n$, if $f\in C_{0,q}^1(\overline{D})$, then*

$$f=\int_{\partial D}f\wedge\Omega_q(W)+\bar{\partial}T_q^Wf+T_{q+1}^W\bar{\partial}f\quad on\quad D.$$

Remark. *If $W = \sum_{j=1}^{n} w_j(\zeta, z)d\zeta_j$ is holomorphic in z, then $\Omega_q(W) = 0$ for $q \geq 1$. In this case, if f is a $\bar\partial$closed $(0, q)$-form, then it holds that $f = \bar\partial(T_q^W f)$.*

In 1965, Hörmander obtained L^2 estimates for solutions of the $\bar\partial$-problem in bounded pseudoconvex domains in $\mathbb{C}^n$. On the other hand, L^p and Hölder estimates for solutions of the $\bar\partial$-problem using the above integral formula have been studied since 1970. We begin with the $\bar\partial$-problem in strictly pseudoconvex domains in $\mathbb{C}^n$.

1. $\bar\partial$-problems in bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary.

Theorem 2. (Henkin[10],Ramirez[19]) *Suppose $D \Subset \mathbb{C}^n$ is strictly pseudoconvex with C^∞ boundary. There are a neighborhood U of ∂D, positive constants δ, c and γ, and a function $g \in C^\infty(U \times D_\delta)$ with the following properties:*

(i) *$g(\zeta, z)$ is holomorphic in z on D_δ.*
(ii) *$g(\zeta, \zeta) = 0 \quad$ for $\zeta \in U$.*
(iii) *$\mathrm{Re}g(\zeta, z) > 0$ for $(\zeta, z) \in U \times D_\delta$ with $r(\zeta) - r(z) + c|\zeta - z|^2 > 0$.*
(iv) *On $\{(\zeta, z) \in U \times D_\delta : |\zeta - z| \leq \gamma\}$ there is a function $A \in C^\infty(U \times D_\delta)$ with $|A(\zeta, z)| \geq \frac{2}{3}$, so that $g = FA$, where F is the Levi polynomial.*

Using Hefer's theorem, there are functions $g_j \in C^\infty(U \times D_\delta)$, with $g_j(\zeta, \cdot) \in \mathcal{O}(D_\delta)$ such that

$$g(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, z)(\zeta_j - z_j) \quad \text{on} \quad U \times D_\delta$$

We define

$$W^{HR} = \sum_{j=1}^{n} \frac{g_j(\zeta, z)}{g(\zeta, z)} d\zeta_j.$$

Then W^{HR} is called the Henkin-Ramirez generating form. Using the above Henkin-Ramirez generating form, the following theorem was obtained(cf. Henkin[10], Kerzman[13], Lieb[16], Henkin-Romanov[11], Grauert-Lieb[9], Range-Siu[25]).

Theorem 3. *Let $D \Subset \mathbb{C}^n$ be strictly pseudoconvex with smooth boundary. For $1 \leq q \leq n$, there are linear operators*

$$\hat S_q : L^1_{0,q}(D) \to L^1_{0,q-1}(D)$$

and a constant C with the following properties:

(i) *$\|\hat S_q f\|_{L^p(D)} \leq C\|f\|_{L^p(D)} \quad$ for $\quad 1 \leq p \leq \infty$.*
(ii) *$\|\hat S_q\|_{\Lambda_{1/2}(D)} \leq C\|f\|_{L^\infty(D)}$.*
(iii) *For $k = 0, 1, 2, \cdots$, if $f \in L^1_{0,q}(D) \cap C^k(D)$, then $\hat S_q f \in C^k_{0,q-1}(D)$.*
(iv) *If $f \in C^1_{0,q}(D) \cap L^1_{0,q}(D)$ and $\bar\partial f = 0$, then $\bar\partial(\hat S_q f) = f$ on D.*

Krantz[14] obtained the optimal Lipschitz and L^p estimates for $\bar{\partial}$ in strictly pseudoconvex domains:

Theorem 4. *Let D be a bounded strictly pseudoconvex domain with C^5 boundary. Let $A^{\infty}_{(0,1)}(D)$ be the space of all $\bar{\partial}$-closed $(0,1)$-forms f whose coefficients are C^{∞} in $\overline{D}$. Then there is a linear operator*

$$H : A^{\infty}_{(0,1)}(D) \to C^{\infty}(D)$$

satisfying $\bar{\partial} H f = f$. Moreover Hf satisfies

(i) $||Hf||_{L^{(2n+2)/(2n+1)-\varepsilon}} \leq A_{\varepsilon}||f||_{L^1}$ *for small enough* $\varepsilon > 0$
(ii) *if* $1 < p < 2n+2$, *then* $||Hf||_{L^q} \leq A_p||f||_{L^p}$, *where* $\frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2}$
(iii) *if* $2n+2 < p \leq \infty$, *then* $||Hf||_{\Lambda_{\alpha}} \leq A_p||f||_{L^p}, \alpha = \frac{1}{2} - \frac{n+1}{p}$.

For $i \in \{1, \cdots, N\}$, we denote by D_i a strictly pseudoconvex open sets in $\mathbb{C}^n$ with C^2 boundary. Let ρ_i be a defining function for D_i. For sufficiently small $\delta_i > 0$ we denote $V_i^{\delta} = \{-\delta < \rho_i(z) < \delta\}$. we assume that for $1 \leq i_1 < i_2 < \cdots < i_l \leq N$, $d\rho_{i_1}, d\rho_{i_2}, \cdots, d\rho_{i_l}$ are $\mathbb{R}$-linearly independent at all points of $V_{i_1}^{\delta} \cap V_{i_2}^{\delta} \cap \cdots \cap V_{i_l}$. We set $D = \cap_{i=1}^{N} D_i$. Then Menini[17] proved the following:

Theorem 5. *Let $f \in L^p_{0,q}(D)(1 \leq q \leq n, 1 \leq p \leq \infty)$ be $\bar{\partial}$-closed. Then there exists a kernel K such that if one defines on D, $T_q f(z) = c_{q,n} \int_D f(\zeta) \wedge K(\zeta, z)$ then $\bar{\partial}(T_q f) = f$. Moreover*

(i) *for* $1 \leq p < \infty$,

$$T_q : L^p_{(0,q)}(D) \to L^r_{0,q-1}(D)$$

is a bounded linear operator where $\frac{1}{r} = \frac{1}{p} + \frac{1}{1+\eta}$, $0 \leq \eta < \frac{1}{2n-1+2\inf(N_0,n-1)}$, *where N_0 is the maximal number of the common intersections,*

(ii) *for* $p = \infty$

$$T_q : L^p_{(0,q)}(D) \to \Lambda^{1/2-\varepsilon}_{(0,q-1)}(D)$$

is a bounded linear operator for any $\varepsilon > 0$.

2. $\bar{\partial}$-problems in q-convex domains in a complex manifold.

Theorem 6. (Fischer-Lieb[7]) *Let X be a complex manifold and let $D \Subset X$ be a strongly q-convex domain (in the sense of Andreotti-Grauert) with C^3 boundary. Then there exists a constant K with the following properties:*

For each $\bar{\partial}$-closed $(0,r)$-form β on D with $r \geq q$ there exists a $(0,r-1)$-form α on D with $\bar{\partial}\alpha = \beta$ and $|\alpha| \leq K|\beta|$.

Let X be an n-dimensional complex manifold. $D \Subset X$ is called a strictly q-convex C^2 intersection if there exists a finite number of real C^2 functions $\rho_1, \cdots, \rho_N$ in a neighborhood U of $\overline{D}$ such that

$$D = \{z \in U : \rho_j(z) < 0 \quad \text{for} \quad 1 \leq j \leq N\}$$

and the following condition is filfilled: if $z \in \partial D$ and $1 \leq k_1 < \cdots < k_l \leq N$ with $\rho_{k_1}(z) = \cdots = \rho_{k_l}(z) = 0$, then

$$d\rho_{k_1}(z) \wedge \cdots \wedge d\rho_{k_l}(z) \neq 0$$

and, for all $\lambda_1, \cdots, \lambda_l \geq 0$ with $\lambda_1 + \cdots + \lambda_l = 1$, the Levi form at z of the function

$$\lambda_1 \rho_{k_1} + \cdots + \lambda_l \rho_{k_l}$$

has at least $q+1$ positive eigenvalues. D is called completely q-convex if there exists a real C^2 function φ on D whose Levi form has at least $q+1$ positive eigenvalues at each point in D and such that

$$\{z \in D : \varphi(z) < C\} \Subset D \quad \text{for all} \quad C > 0.$$

Let E be a holomorphic vector bundle over X. Denote by $B^{\beta}_{n,r}(D,E), \beta \geq 0, r = 0,1,\cdots,n$, the Banach space of E-valued continuous (n,r)-forms f on D such that

$$\sup_{z \in D} ||f(z)|| [\text{dist}(z, \partial D)]^{\beta} < \infty,$$

and denote by $C^{\alpha}_{n,r}(\overline{D},E), 0 \leq \alpha \leq 1, r = 0,1,\cdots,n$, the Banach space of E-valued (n,r)-forms which are Hölder continuous with exponent α on $\overline{D}$. In this setting, Laurent-Thiébaut-Leiterer[15] proved the following:

Theorem 7. *Let $D \Subset X$ be a strictly q-convex C^2 intersection and completely q-convex. Then:*

(i) *If $0 \leq \beta < \frac{1}{2}$, then there exist linear operators*

$$T_r : B^{\beta}_{n,r}(D,E) \cap \ker d \to \cap_{0<\varepsilon\leq 1/2-\beta} C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D},E),$$

$n - q \leq r \leq n$, which are compact as operators from $B^{\beta}_{n,r}(D,E) \cap \ker d$ to each $C^{1/2-\beta-\varepsilon}_{n,r-1}(\overline{D},E), 0 < \varepsilon \leq 1/2-\beta$, and such that

$$dT_r f = f$$

for all $n-q \leq r \leq n$ *and* $f \in B^{\beta}_{n,r}(D,E) \cap \ker d$.

(ii) *If* $1/2 \leq \beta < 1$, *then there exist linear operators*

$$T_r : B^{\beta}_{n,r}(D,E) \cap \ker d \to \cap_{0<\varepsilon} B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E),$$

$n-q \leq r \leq n$, *which are compact as operators from* $B^{\beta}_{n,r}(D,E) \cap \ker d$ *to each* $B^{\beta+\varepsilon-1/2}_{n,r-1}(D,E), \varepsilon > 0$, *and such that*

$$dT_r f = f$$

for all $n-q \leq r \leq n$ *and* $f \in B^{\beta}_{n,r}(D,E) \cap \ker d$.

3. $\bar{\partial}$-problems in bounded weakly pseudoconvex domains in $\mathbb{C}^n$.

In the case of weakly pseudoconvex domains there are several results in $\mathbb{C}^2$.

Thoerem 8. (Range[21]) *Let* $D \subset \mathbb{C}^2$ *be a bounded convex domain with real analytic boundary. Then there are positive constants* α *and* K *such that for every bounded* $\bar{\partial}$*-closed* $f \in C^1_{0,1}(D)$ *there is* $u \in C^1(D)$ *such that* $\bar{\partial} u = f$ *and*

$$|u(z) - u(z')| \leq K||f||_{L^\infty(D)}|z-z'|^\alpha, \quad z, z' \in D.$$

Theorem 9. (Show[26]) *Let* D *be a pseudoconvex domain in* $\mathbb{C}^2$ *of uniform strict type* m. *Let* f *be a continuous (0,1)-form on* $\overline{D}$ *and* $\bar{\partial} f = 0$, *then there exists a function* $u \in \Lambda_{1/m}(\overline{D})$ *such that* $\bar{\partial} u = f$ *and* u *satisfies the following estimates:*

(i) $||u||_{L^1(D)} \leq c(||f||_{L^1(D)} + ||f||_{L^1(\partial D)})$,

(ii) *if* $p=1$, *then* $||u||_{L^{(m+2)/(m+1)-\varepsilon}(\partial D)} \leq c||f||_{L^1(\partial D)}$ *for every small* $\varepsilon > 0$,

(iii) *if* $1 < p < m+2$, *then* $||u||_{L^q(\partial D)} \leq c_p||f||_{L^p(\partial D)}$ *where* $\frac{1}{q} = \frac{1}{p} - \frac{1}{m+2}$,

(iv) *if* $p = m+2$, *then* $||u||_{L^q(\partial D)} \leq c_p||f||_{L^p(\partial D)}$ *for all* $q < \infty$,

(v) *if* $m+2 < p \leq \infty$, *then* $||u||_{\Lambda_{1/m-(m+2)/mp}(\partial D)} \leq c_p||f||_{L^p(\partial D)}$,

(vi) $||u||_{\Lambda^p_{1/m}(\partial D)} \leq c_p||f||_{L^p(\partial D)}$ *for every* $1 \leq p \leq \infty$.

Theorem 10. (Range[23]) *Let* D *be a smoothly bounded pseudoconvex domain in* $\mathbb{C}^2$ *of finite type* m, *and let* $f \in C^1_{0,1}(\overline{D})$ *be* $\bar{\partial}$*-closed. Then for every* $\eta > 0$ *there is a solution* $u^{(\eta)}$ *of* $\bar{\partial} u = f$ *on* D *which satisfies*

$$|u^{(\eta)}(z) - u^{(\eta)}(w)| \leq C_\eta ||f||_{L^\infty}|z-w|^{(1/m)-\eta}$$

for $z, w \in D$.

Theorem 11. (Polking[18]) *Let $D \Subset \mathbb{C}^2$ be convex with C^2 boundary. Then there is an integral solution operator T for $\bar{\partial}$ on D such that $\|Tf\|_{L^p(D)} \le C_p\|f\|_{L^p(D)}$ for all $1 < p < \infty$.*

Theorem 12. (Range[24]) *Let $D \Subset \mathbb{C}^2$ be convex with C^2 boundary. Then there is an integral solution operator T for $\bar{\partial}$ on D such that*

(i) $|Tf|_{\Lambda_\alpha(D)} \le C_\alpha|f|_{\Lambda_\alpha(D)}$ *for all f with $\bar{\partial}f = 0$ and all $\alpha > 0$.*
(ii) $\|Tf\|_{BMO(D)} \le C\|f\|_{L^\infty(D)}$.

Now we study the uniform and L^p estimates for solutions of the $\bar{\partial}$-problem in pseudoconvex domains which may be of infinite type.

Let $\Psi \in C^2([0,1])$ be a real valued function satisfying

(A) $\Psi(0) = 0$ and $\Psi(1) = 1$.
(B) $\Psi'(t) > 0, \quad 0 < t \le 1$.
(C) $\Psi'(t) + t\Psi''(t) > 0, \quad 0 < t \le 1$.
(D) There exists $\tau \in (0,1)$ such that $\Psi''(t) > 0, \quad 0 < t < \tau$.

Define

$$D_\Psi = \{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \cdots, n, \sum_{j=1}^{n-1} |z_j|^2 + \Psi(|z_n|^2) < 1\}.$$

For $\alpha > 0$, define $\Psi_\alpha(t) = e\exp(-1/t^\alpha)$. Then Ψ_α satisfies all conditions (A)-(D). In this case the domain D_{Ψ_α} is not of finite type.

Theorem 13. (Adachi-Cho[2]) *Let $f \in L^p_{0,q}(D_\Psi)$, $1 \le p \le \infty$, be $\bar{\partial}$-closed. If $\int_0^1 |\log \Psi(s)|s^{-\frac{1}{2}}ds < \infty$, then there is a solution u of $\bar{\partial}u = f$ on D_Ψ such that*

$$||u||_{L^p(D_\Psi)} \le c(p)||f||_{L^p(D_\Psi)}.$$

where the constant $c(p)$ is independent of f.

Remark. *In case $n = 2$ and $p = \infty$, Theorem 13 was obtained by Verdera[28].*

4. $\bar{\partial}$- problems in ellipsoids.

Define

$$D_1 = \{z \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^{m_i} < 1\},$$

$$D_2 = \{z = (z_1, \cdots, z_n) : \sum_{i=1}^{n} (x_i^{l_i} + y_i^{m_i}) < 1, z_j = x_j + iy_j\},$$

where m_i, l_i are positive even integers. We set

$$k_1 = \sup_{1\le i\le n} \{m_i\}, \quad k_2 = \sup_{1\le i\le n} \{\inf\{l_i, m_i\}\}.$$

We have the following:

Theorem 14. (Range[20]) *For each $\alpha < 1/k_1$, there exists a constant C_α such that for every bounded, $\bar\partial$-closed (0,1)-form f on D_1, there exists a α-Hölder continuous function u on D_1 such that $\bar\partial u = f$ and $||u||_{\Lambda_\alpha(D_1)} \le C_\alpha ||f||_{L^\infty(D_1)}$.*

Theorem 15. (Diederich-Fornaess-Wiegerinck[5]) *There exists a constant C such that for every bounded, $\bar\partial$-closed (0,1)-form f on D_2, there exists a $\alpha = 1/k_2$-Hölder continuous function u on D_2 such that $\bar\partial u = f$ and $||u||_{\Lambda_\alpha(D_2)} \le C||f||_{L^\infty(D_2)}$.*

Remark. *Diederich, Fornaess and Wiegerinck pointed out in their paper that Theorem 14 is also true in case $\alpha = 1/k_1$.*

Theorem 16. (Chen-Krantz-Ma[4]) *Let D_1 be the complex ellipsoid defined above. Then for every $\bar\partial$-closed $(0,1)$-form f with coefficients in $L^p(D_1)$, there exists a function u on D_1 such that $\bar\partial u = f$, and u satisfies the following estimates:*

(i) *if $p = 1$, then $\mu\{|u| > t\} \le C\{||f||_{L^1(D_1)}\frac{1}{t}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{k_1 n+2}{k_1 n+1}$,*
(ii) *if $1 < p < k_1 n + 2$, then $||u||_{L^q(D_1)} \le C_p||f||_{L^p(D_1)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{k_1 n+2}$,*
(iii) *if $p = k_1 n + 2$, then $||u||_{L^q(D_1)} \le C_p||f||_{L^p(D_1)}$ for all $q < \infty$.*
(iv) *if $p > k_1 n + 2$, then $||u||_{\Lambda_\alpha(D_1)} \le C_p||f||_{L^p(D_1)}$, where $\alpha = \frac{1}{k_1} - (n + \frac{2}{k_1})\frac{1}{p}$.*

We give the results obtained by Fleron[8]. Ho[12] obtained similar results in the case where D is a complex ellipsoid.

Theorem 17. (Fleron[8]) *Let $1 \le q \le n-1$. Let D be a real or complex ellipsoid. Suppose that Δ^q is the maximal order of contact of the boundary of the ellipsoid D with q-dimensional complex linear spaces. Then there are linear operators $T_q : C_{(0,q)}(\overline{D}) \to C_{(0,q-1)}(D)$ satisfying the following:*

(i) *if $f \in C^1_{(0,q)}(\overline{D})$ and $\bar\partial f = 0$, then $\bar\partial(T_q f) = f$ on D,*
(ii) *there is a constant $c >$ such that $|T_q f(z) - T_q f(z')| \le c||f||_{L^\infty(D)}|z - z'|^{\frac{1}{\Delta^q}}$ for $z, z' \in D$.*

Now we give the following optimal L^p estimates for solutions of the $\bar\partial$-problem in ellipsoids.

Theorem 18. (Adachi[1]) *Let m be the maximal order of contact of the boundary of the complex ellipsoid D with q-dimensional complex linear subspaces. Let $p \ge 1$. Then for every $\bar\partial$-closed $(0,q)$-form f with coefficients in $L^p(D)$, there exists a function u on D such that $\bar\partial u = f$, and u satisfies the following estimates:*

(i) *if $p = 1$, then $\mu\{|u| > t\} \le C\{||f||_{L^1(D)}\frac{1}{t}\}^\lambda$ for all $t > 0$, where $\lambda = \frac{mn+2}{mn+1}$,*
(ii) *if $1 < p < mn + 2$, then $||u||_{L^s(D)} \le C_p||f||_{L^p(D)}$, where $\frac{1}{s} = \frac{1}{p} - \frac{1}{mn+2}$,*
(iii) *if $p = mn + 2$, then $||u||_{L^s(D)} \le C_p||f||_{L^p(D)}$ for all $s < \infty$,*
(iv) *if $p > mn + 2$, then $||u||_{\Lambda_\alpha(D)} \le C_p||f||_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.*

Theorem 19. (Adachi[1]) *Let m be the maximal order of contact of the boundary of the real ellipsoid D with q-dimensional complex linear subspaces. Let $p \geq 1$. Then for every $\bar{\partial}$-closed $(0,q)$-form f with coefficients in $L^p(D)$, there exists a function u on D such that $\bar{\partial}u = f$ and u satisfies the following estimates:*

(i) *if $p = 1$, then $||u||_{L^{\gamma-\varepsilon}(D)} \leq c||f||_{L^1(D)}$, where $\gamma = \frac{mn+2}{mn+1}$ and ε is any small number,*
(ii) *if $1 < p < mn+2$, then $||u||_{L^s(D)} \leq c||f||_{L^p(D)}$, where $s < q_0$ and q_0 satisfies $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{mn+2}$,*
(iii) *if $p = mn + 2$, then $||u||_{L^s(D)} \leq C_p||f||_{L^p(D)}$ for all $s < \infty$,*
(iv) *if $p > mn + 2$, then $||u||_{\Lambda_\alpha(D)} \leq C_p||f||_{L^p(D)}$, where $\alpha = \frac{1}{m} - (n + \frac{2}{m})\frac{1}{p}$.*

5. Applications of the $\bar{\partial}$-problem.

A. Uniform approximation of holomorphic functions.

Theorem 20. (Kerzman[13]) *Let $D \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. There exists an open set $\hat{D} \Subset \mathbb{C}^n, D \subset \overline{D} \Subset \hat{D}$, which has the following properties:*

(a) *Any continuous function $u : \overline{D} \to \mathbb{C}$ which is holomorphic in D can be uniformly approximated on $\overline{D}$ by holomorphic functions $\hat{u}$ defined on $\hat{D}$.*
(b) *Let $u : D \to \mathbb{C}$ be holomorphic and assume $u \in L^p(D), 1 \leq p \leq \infty$. Then there exists a sequence of holomorphic functions $\hat{u}_n : \hat{D} \to \mathbb{C}$ such that* (b_1), (b_2) and (b_3) *below hold:*
(b_1) *$\hat{u}_n \to u$ uniformly on compact subsets of D when $n \to \infty$,*
(b_2) $||\hat{u}_n||_{L^p(D)} \leq K||u||_{L^p(D)}, \quad 1 \leq p \leq \infty,$
(b_3) *if $p < \infty$, then $||\hat{u}_n - u||_{L^p(D)} \to 0$ when $n \to \infty$, where K is independent of n, p and u.*

B. Vanishing cohomology theorems.

Theorem 21. (Kerzman[13], Lieb[16]) *Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. Let $\mathcal{F}$ and $\mathcal{B}$ be the sheaf of germs of holomorphic functions in D which are continuous on $\overline{D}$ and the sheaf of germs of holomorphic functions in D which are bounded in D, respectively. Then we have*

$$H^q(\overline{D}, \mathcal{F}) = H^q(\overline{D}, \mathcal{B}) = 0 \quad \text{for} \quad q \geq 1.$$

C. The Poincaré-Lelong equation.

Theorem 22. (Show[27]) *Let D be a real ellipsoid in $\mathbb{C}^n$. Given any analytic variety of complex dimension* (n-1) *such that M is the zero sets of an analytic function on D of finite area, there exists a function h belonging to the Nevanlinna class such that M is the zero sets of h.*

Theorem 23. (Arlebrink[3]) *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with C^3 boundary. If X is a positive divisor of D with finite area and the canonical cohomology class of X in $H^2(D, \mathbb{Z})$ is zero, then there exists a bounded holomorphic function that defines X.*

References

1. K. Adachi, *$\bar{\partial}$ problem in ellipsoid*, (preprint).
2. K. Adachi and H. R. Cho, *$L^p(1 \leq p \leq \infty)$ estimates for $\bar{\partial}$ on a certain pseudoconvex domain in $\mathbb{C}^n$*, Nagaya Math. J. **148** (1997), 127–136.
3. J. Arlebrink, *Zeors of bounded holomorphic functions in strictly pseudoconvex domains in $\mathbb{C}^n$*, Ann. Inst. Fourrier, Grenoble **43(2)** (1993), 437–458.
4. Z. Chen, S. G. Krantz and D. Ma, *Optimal L^p estiamtes for the $\bar{\partial}$ estiamtes for the $\bar{\partial}$-equation on complex ellipsoids in $\mathbb{C}^n$*, Manuscripta Math. **80** (1993), 131–149.
5. K. Diederich, J. E. Fornaess and J. Wiegerinck, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids*, Manuscripta Math. **56** (1986), 399–417.
6. C. Fefferman and J. J. Kohn, *Hölder estimates on domains in two complex dimensions and on three dimensional CR manifolds*, Adv. Math. **69** (1988), 233–303.
7. W. Fischer and I. Lieb, *Lokale Kerne und beschränkte Lösungen für den $\bar{\partial}$-Operator auf q-konvexen Gebieten*, Math. Ann. **208** (1974), 249-265.
8. J. F. Fleron, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids and their compliments via order of contact*, Proc. Amer. Math. Soc. **124(10)** (1996), 3193–3202.
9. H. Grauert and I. Lieb, *Das Ramirezsche Integral und die Gleichung $\bar{\partial}u = f$ im Bereich der beschrankten Formen*, Rice Univ. Studies **56** (1970), 29–50.
10. G. M. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications*, Math. USSR-Sb. **11** (1970), 273–282.
11. G. M. Henkin and A. V. Romanov, *Exact Hölder estimates of solutions of the $\bar{\partial}$-equation*, Math. USSR-Izv. **5** (1971), 1180–1192.
12. L. H. Ho, *Hölder estimates for localsolutions for $\bar{\partial}$ on a class of nonpseudoconvex domains*, Rocky Mountain J. Math. **23(2)** (1993), 593–607.
13. N. Kerzman, *Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301–379.
14. S. G. Krantz, *Optimal Lipschitz and L^p regularity for the euation $\bar{\partial}u = f$ on strongly pseudoconvex domains*, Math. Ann. **219** (1976), 233-260.
15. C.Laurent-Thiébaut and J. Leierer, *Uniform estimates for the Cauchy-Riemann equation on q-convex wedges*, Ann. Inst. Fourier, Grenoble **43(2)** (1993), 383–436.
16. I. Lieb, *Die Cauchy-Riemannschen Differentialgleichungen auf streng pseudokonvexen Gebieten: Beschränkte Rösungen*, Math. Ann. **190** (1970), 6–44.
17. C. Menini, *Estimations pour la resolution du $\bar{\partial}$ sur une intersection d'ouverts strictment pseudoconvexes*, Math. Z. **225** (1997), 87–93.
18. J. Polking, *The Cauchy Riemann equations on convex sets*, Proc. Symp. Pure Math. **52(3)** (1991), 309–322.
19. E. Ramirez, *Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis*, Math. Ann. **184** (1970), 172–187.
20. R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Proc. Int. Conf. Cortona (1977).

21. ______, *Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbb{C}^2$ with real analytic boundary*, Proc. Symp. Pure Math. (1977).
22. ______, *Holomorphic functions and integral representations in several complex variables*, Berlin Heiderberg New York (1986).
23. ______, *Integral kernels and Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$*, Math. Ann. **288** (1990), 65–74.
24. ______, *On Hölder and BMO estimates for $\bar{\partial}$ on convex domains in $\mathbb{C}^2$*, J. Geom. Anal. **2(6)** (1992), 575–584.
25. R. M. Range and Y. T. Siu, *Uniform estimates for the $\bar{\partial}$-euation on domains with piecewise smooth strictly pseudoconvex boundaries*, Math. Ann. **206** (1973), 325–354.
26. M. C. Show, *Prescribing zeros of functions in the Nevanlinna class on weakly pseudoconvex domains in $\mathbb{C}^2$*, Trans. Amer. Math. Soc. **313(1)** (1989), 407–418.
27. ______, *Optimal Hölder and L^p estimates for $\bar{\partial}_b$ on the boundaries of real ellipsoids in $\mathbb{C}^n$*, Trans. Amer. Math. Soc. **324(1)** (1991), 213–234.
28. J. Verdera, *L^∞-continuity of Henkin operators solving $\bar{\partial}$ in certain weakly pseudoconvex domains of $\mathbb{C}^2$*, Proc. Royal Soc. Edinburgh **99A** (1984), 25–33.